

On Edge-Disjoint Pairs Of Matchings*

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Abstract

For a graph G , consider the pairs of edge-disjoint matchings whose union consists of as many edges as possible. Let H be the largest matching among such pairs. Let M be a maximum matching of G . We show that $5/4$ is a tight upper bound for $|M|/|H|$.

Keywords: matching, maximum matching, pair of edge-disjoint matchings

We consider finite, undirected graphs without multiple edges or loops. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph G , respectively. The cardinality of a maximum matching of a graph G is denoted by $\nu(G)$.

For a graph G define $B_2(G)$ as follows:

$$B_2(G) \equiv \{(H, H') : H, H' \text{ are edge-disjoint matchings of } G\},$$

and set:

$$\lambda_2(G) \equiv \max\{|H| + |H'| : (H, H') \in B_2(G)\}.$$

Define:

$$\alpha_2(G) \equiv \max\{|H|, |H'| : (H, H') \in B_2(G) \text{ and } |H| + |H'| = \lambda_2(G)\},$$
$$M_2(G) \equiv \{(H, H') : (H, H') \in B_2(G), |H| + |H'| = \lambda_2(G), |H| = \alpha_2(G)\}.$$

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It is clear that $\alpha_2(G) \leq \nu(G)$ for all G . By Mkrtchyan's result [4], reformulated as in [2], if G is a matching covered tree then the inequality turns to an equality. Note that a graph is said to be matching covered (see [5]) if its every edge belongs to a maximum matching (not necessarily a perfect matching as it is usually defined, see e.g. [3]).

The aim of this paper is to obtain a tight upper bound for $\frac{\nu(G)}{\alpha_2(G)}$. We prove that $\frac{5}{4}$ is an upper bound for $\frac{\nu(G)}{\alpha_2(G)}$, and exhibit a family of graphs which shows that $\frac{5}{4}$ can not be replaced by any smaller constant. Terms and concepts that we do not define can be found in [1, 3, 6].

Let A and B be matchings of a graph G .

Definition. A path (or an even cycle) e_1, e_2, \dots, e_l ($l \geq 1$) is called A - B alternating if the edges with odd indices belong to $A \setminus B$ and others to $B \setminus A$, or vice-versa.

Definition. A - B alternating path P is called maximal if there is no other A - B alternating path that contains P as a proper subpath.

The sets of A - B alternating cycles and maximal alternating paths are denoted by $C(A, B)$ and $P(A, B)$, respectively.

The set of the paths from $P(A, B)$ that have even (odd) length is denoted by $P_e(A, B)$ ($P_o(A, B)$).

The set of the paths from $P_o(A, B)$ starting from an edge of A (B) is denoted by $P_o^A(A, B)$ ($P_o^B(A, B)$).

Note that every edge $e \in A \cup B$ either belongs to $A \cap B$ or lies on a cycle from $C(A, B)$ or lies on a path from $P(A, B)$.

Moreover,

Property 1 (a) if $F \in C(A, B) \cup P_e(A, B)$ then A and B have the same number of edges that belong to F ,

(b) if $P \in P_o^A(A, B)$ then the difference between the numbers of edges that lie on P and belong to A and B is one.

These observations immediately imply:

Property 2 If A and B are matchings of a graph G then

$$|A| - |B| = |P_o^A(A, B)| - |P_o^B(A, B)|.$$

Berge's well-known theorem states that a matching M of a graph G is maximum if and only if G does not contain an M -augmenting path [1,3,6]. This theorem immediately implies:

Property 3 If M is a maximum matching and H is a matching of a graph G then

$$P_o^H(M, H) = \emptyset,$$

and therefore, $|M| - |H| = |P_o^M(M, H)|$.

The proof of the following property is similar to the one of property 3:

Property 4 If $(H, H') \in M_2(G)$ then $P_o^{H'}(H, H') = \emptyset$.

Let G be a graph. Over all $(H, H') \in M_2(G)$ and all maximum matchings M of G , consider the pairs $((H, H'), M)$ for which $|M \cap H|$ is maximized. Among these, choose a pair $((H, H'), M)$ such that $|M \cap H'|$ is maximized.

From now on H, H' and M are assumed to be chosen as described above. For this choice of H, H' and M , consider the paths from $P_o^M(M, H)$ and define M_A and H_A as the sets of edges lying on these paths that belong to M and H , respectively.

Lemma 1 $C(M, H) = P_e(M, H) = P_o^H(M, H) = \emptyset$.

Proof. Property 3 implies $P_o^H(M, H) = \emptyset$. Let us show that $C(M, H) = P_e(M, H) = \emptyset$. Suppose that there is $F_0 \in C(M, H) \cup P_e(M, H)$. Define:

$$M' \equiv [M \setminus E(F_0)] \cup [H \cap E(F_0)].$$

Consider the pair $((H, H'), M')$. Note that M' is a maximum matching, and

$$|H \cap M'| > |H \cap M|,$$

which contradicts $|H \cap M|$ being maximum. ■

Corollary 1 $M \cap H = M \setminus M_A = H \setminus H_A$.

Lemma 2 Each edge of $M_A \setminus H'$ is adjacent to two edges of H' .

Proof. Let e be an arbitrary edge from $M_A \setminus H'$. Note that $e \in M$, $e \notin H$, $e \notin H'$. Now, if e is not adjacent to an edge of H' , then $H \cap (H' \cup \{e\}) = \emptyset$ and

$$|H| + |H' \cup \{e\}| > |H| + |H'| = \lambda_2(G),$$

which contradicts $(H, H') \in M_2(G)$.

On the other hand, if e is adjacent to only one edge $f \in H'$, then consider the pair (H, H'') , where $H'' \equiv (H' \setminus \{f\}) \cup \{e\}$. Note that

$$H \cap H'' = \emptyset, \quad |H''| = |H'|$$

and

$$|H'' \cap M| > |H' \cap M|,$$

which contradicts $|H' \cap M|$ being maximum. ■

Lemma 3 $C(M_A, H') = P_e(M_A, H') = P_o^{M_A}(M_A, H') = \emptyset$.

Proof. First of all, let us show that $C(M_A, H') = P_e(M_A, H') = \emptyset$. For the sake of contradiction, suppose that there is $F_0 \in C(M_A, H') \cup P_e(M_A, H')$. Define:

$$H'' \equiv [H' \setminus E(F_0)] \cup [M_A \cap E(F_0)].$$

Consider the pair of matchings (H, H'') . Note that due to the definition of an alternating path we have $M_A \cap H = \emptyset$, therefore

$$H \cap H'' = \emptyset,$$

$$|H| + |H''| = |H| + |H'| = \lambda_2(G)$$

(see (a) of property 1).

Thus $(H, H'') \in M_2(G)$ and

$$|H'' \cap M| > |H' \cap M|,$$

which contradicts $|H' \cap M|$ being maximum.

On the other hand, the end-edges of a path from $P_o^{M_A}(M_A, H')$ are from M_A and are adjacent to only one edge from H' contradicting lemma 2. Therefore, $P_o^{M_A}(M_A, H') = \emptyset$. ■

Lemma 4 $|H'| = |P_o^{H'}(M_A, H')| + |H_A| + \nu(G) - \alpha_2(G)$.

Proof. Due to property 2

$$|H'| - |M_A| = |P_o^{H'}(M_A, H')| - |P_o^{M_A}(M_A, H')|,$$

and due to (b) of property 1 and property 3

$$|M_A| - |H_A| = |P_o^M(M, H)| = |M| - |H| = \nu(G) - \alpha_2(G).$$

By lemma 3 $P_o^{M_A}(M_A, H') = \emptyset$, therefore,

$$|H'| = |P_o^{H'}(M_A, H')| + |M_A| = |P_o^{H'}(M_A, H')| + |H_A| + \nu(G) - \alpha_2(G).$$

■

Lemma 5 Let $P \in P_o(M, H)$ and assume that $P = m_1, h_1, m_2, \dots, h_{l-1}, m_l$, $l \geq 1, m_i \in M, 1 \leq i \leq l, h_j \in H, 1 \leq j \leq l-1$. Then $l \geq 3$ and $\{m_1, m_l\} \subseteq H'$.

Proof. Let us show that $m_1, m_l \in H'$. If $l = 1$ then $P = m_1$, $m_1 \in M \setminus H$, and m_1 is not adjacent to an edge from H as P is maximal. Thus, $m_1 \in H'$ as otherwise we could enlarge H by adding m_1 to it which contradicts $(H, H') \in M_2(G)$. Thus suppose that $l \geq 2$. Let us show that $m_1 \in H'$. If $m_1 \notin H'$ then define

$$H_1 \equiv (H \setminus \{h_1\}) \cup \{m_1\}.$$

Clearly, H_1 is a matching, $H_1 \cap H' = \emptyset$ and $|H_1| = |H|$, which means that $(H_1, H') \in M_2(G)$. But

$$|H_1 \cap M| > |H \cap M|,$$

which contradicts $|H \cap M|$ being maximum. Similarly, it could be shown that $m_l \in H'$.

Now let us show that $l \geq 3$. Due to property 4, $P_o^{H'}(H, H') = \emptyset$. Thus there is i , $1 \leq i \leq l$, such that $m_i \in M \setminus H'$. Since $\{m_1, m_l\} \subseteq H'$, we have $l \geq 3$. ■

Corollary 2 $|H_A| \geq 2(\nu(G) - \alpha_2(G))$.

Proof. Due to lemma 5 every path $P \in P_o(M, H)$ has length at least five, therefore it contains at least two edges from H . Due to property 3, there are

$$|P_o(M, H)| = |P_o^M(M, H)| = \nu(G) - \alpha_2(G)$$

paths from $P_o(M, H)$, therefore

$$|H_A| \geq 2(\nu(G) - \alpha_2(G)).$$

■

Corollary 3 *Every vertex lying on a path from $P(M, H) = P_o^M(M, H)$ is incident to an edge from H' .*

Proof. Suppose w is a vertex lying on a path from $P(M, H) = P_o^M(M, H)$ and assume that e is an edge from M_A incident to the vertex w . Clearly, if $e \in H'$ then the corollary is proved therefore we may assume that $e \notin H'$. Note that $e \in M_A \setminus H'$ therefore due to lemma 2 e is adjacent to two edges from H' . Thus w is incident to an edge from H' . ■

Let Y denote the set of the paths from $P(H, H')$ starting from the end-edges of the paths from $P_o^M(M, H)$. Note that Y is well-defined since due to lemma 5 these end-edges belong to H' . According to property 4, $Y \subseteq P_e(H, H')$, thus the set of the last edges of the paths from Y is a subset of H . Let us denote it by H_Y .

Lemma 6

(a) $|Y| = 2(\nu(G) - \alpha_2(G))$ and the length of the paths from Y is at least four,

(b) $|P_o^{H'}(M_A, H')| \geq \nu(G) - \alpha_2(G)$.

Proof. (a) Due to property 4, all end-edges of the paths from $P_o^M(M, H)$ lie on different paths from Y . Therefore $|Y| = 2|P_o^M(M, H)| = 2(\nu(G) - \alpha_2(G))$.

Since the edges from H_Y are adjacent to only one edge from H' , we conclude that they do not lie on a path from $P_o^M(M, H)$ (corollary 3). Thus, due to corollary 1, $H_Y \subseteq M \cap H$. Furthermore, as the first two edges of a path from

Y lie on a path from $P_o^M(M, H)$, and the last edge does not, we conclude that its length is at least four.

(b) From $H_Y \subseteq M \cap H$ we get

$$|M \cap H| \geq |H_Y| = |Y| = 2|P_o^M(M, H)| = 2(\nu(G) - \alpha_2(G)).$$

On the other hand, every edge from H_Y is adjacent to an edge from $H' \setminus M$, which is an end-edge of a path from $P_o^{H'}(M_A, H')$, therefore

$$2(\nu(G) - \alpha_2(G)) \leq |M \cap H| \leq 2|P_o^{H'}(M_A, H')|$$

or

$$\nu(G) - \alpha_2(G) \leq |P_o^{H'}(M_A, H')|.$$

■

Theorem For every graph G the inequality $\frac{\nu(G)}{\alpha_2(G)} \leq \frac{5}{4}$ holds.

Proof. Lemma 4, statement (b) of lemma 6 and corollary 2 imply

$$\alpha_2(G) \geq |H'| = |P_o^{H'}(M_A, H')| + |H_A| + \nu(G) - \alpha_2(G) \geq 4(\nu(G) - \alpha_2(G)).$$

Therefore, $\frac{\nu(G)}{\alpha_2(G)} \leq \frac{5}{4}$. ■

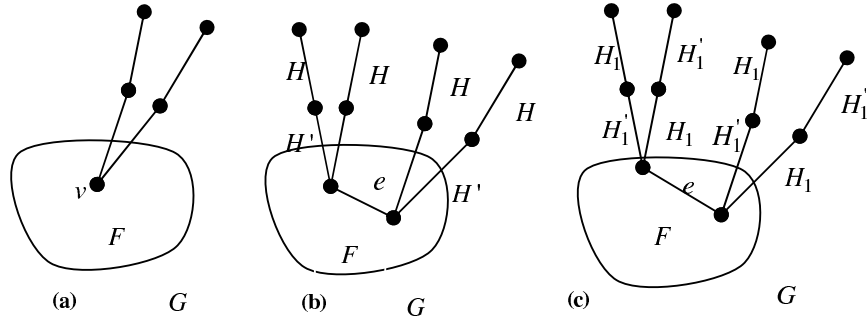


Figure 1: $\frac{\nu(G)}{\alpha_2(G)} = \frac{5}{4}$

Remark 1. We have given a proof of the theorem which is based on the structural lemma 4, statement (b) of lemma 6 and corollary 2. It is not hard to see that the theorem can also be proved directly using only statement (a) of lemma 6. As the length of every path from Y is at least four, there are at least two edges from H' lying on each path from Y , therefore

$$\alpha_2(G) \geq |H'| \geq 2|Y| = 4(\nu(G) - \alpha_2(G)).$$

Remark 2. There are infinitely many graphs G for which

$$\frac{\nu(G)}{\alpha_2(G)} = \frac{5}{4}.$$

In order to construct one, just take an arbitrary graph F containing a perfect matching. Attach to every vertex v of F two paths of length two, as it is shown on the figure 1a.

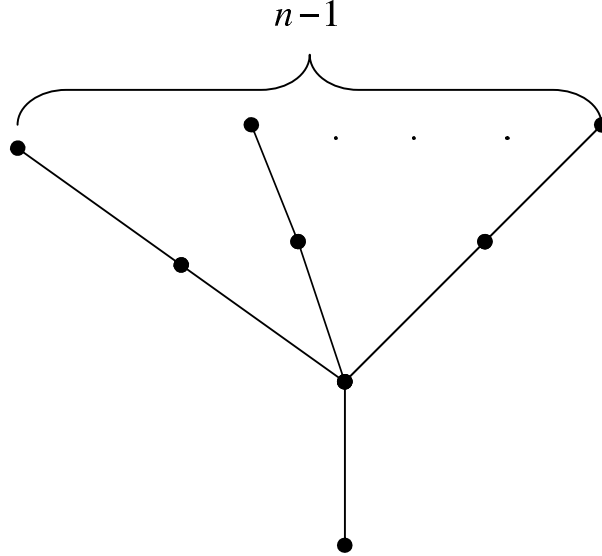


Figure 2: $\frac{\nu(G_n)}{\lambda_2(G_n) - \alpha_2(G_n)} = n$

Let G be the resulting graph. Note that:

$$\nu(G) = \frac{|V(F)|}{2} + 2|V(F)| = \frac{5|V(F)|}{2}.$$

Let us show that for every pair of disjoint matchings (H, H') satisfying $|H| + |H'| = \lambda_2(G)$ and $e \in E(F)$ we have $e \notin H \cup H'$. On the opposite assumption, consider an edge $e \in E(F)$ and a pair (H, H') with $|H| + |H'| = \lambda_2(G)$ and $e \in H \cup H'$. Note that without loss of generality, we may always assume that H and H' contain the edges shown on the figure 1b.

Now consider a new pair of disjoint matchings (H_1, H'_1) obtained from (H, H') as it is shown on figure 1c.

Note that $|H_1| + |H'_1| = 1 + |H| + |H'| > \lambda_2(G)$, which contradicts the choice of (H, H') , therefore $e \notin H \cup H'$ and $\lambda_2(G) = 4|V(F)|$, $\alpha_2(G) = 2|V(F)|$, hence

$$\frac{\nu(G)}{\alpha_2(G)} = \frac{5}{4}.$$

Remark 3. In contrast with the bound $\frac{\nu(G)}{\alpha_2(G)} \leq \frac{5}{4}$, it can be shown that for every positive integer $n \geq 2$ there is a graph G_n such that $\frac{\nu(G_n)}{\lambda_2(G_n) - \alpha_2(G_n)} = n$. Just consider the graph G_n shown on the figure 2.

Note that $\nu(G_n) = n$, $\lambda_2(G_n) = n + 1$, $\alpha_2(G) = n$ hence

$$\frac{\nu(G_n)}{\lambda_2(G_n) - \alpha_2(G_n)} = n.$$

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References

- [1] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
- [2] F. Harary, M.D. Plummer, On the core of a graph, Proc. London Math. Soc. 17 (1967), pp. 305–314.
- [3] L. Lovasz, M.D. Plummer, Matching theory, Ann. Discrete Math. 29 (1986).
- [4] V. V. Mkrtchyan, On trees with a maximum proper partial 0-1 coloring containing a maximum matching, Discrete Mathematics 306, (2006), pp. 456-459.
- [5] V. V. Mkrtchyan, A note on minimal matching covered graphs, Discrete Mathematics 306, (2006), pp. 452-455.
- [6] D. B. West, Introduction to Graph Theory, Prentice-Hall, Englewood Cliffs, 1996.